

# ON SOLUTIONS TO SOME POLYNOMIAL CONGRUENCES IN SMALL BOXES

IGOR E. SHPARLINSKI

ABSTRACT. We use bounds of mixed character sum to study the distribution of solutions to certain polynomial systems of congruences modulo a prime  $p$ . In particular, we obtain nontrivial results about the number of solution in boxes with the side length below  $p^{1/2}$ , which seems to be the limit of more general methods based on the bounds of exponential sums along varieties.

## 1. INTRODUCTION

There is an extensive literature investigating the distribution of solutions to the system congruence

$$(1) \quad F_j(x_1, \dots, x_n) \equiv 0 \pmod{p}, \quad j = 1, \dots, m,$$

$F_j(X_1, \dots, X_n) \in \mathbb{Z}[X_1, \dots, X_n]$ ,  $j = 1, \dots, m$ , in  $m$  variables with integer coefficients, modulo a prime  $p$ , see [4, 5, 8, 11, 12].

In particular, subject some additional condition (related to the so-called  $A$ -number), Fouvry and Katz [5, Corollary 1.5] have given an asymptotic formula for the number of solutions to (1) in a box

$$(x_1, \dots, x_n) \in [0, h - 1]^n$$

for a rather small  $h$ . In fact the limit of the method of [5] is  $h = p^{1/2+o(1)}$ .

Here we consider a very special class of systems of  $s + 1$  polynomial congruences

$$(2) \quad x_1 \dots x_n \equiv a \pmod{p},$$

and

$$(3) \quad c_{1,j}x_1^{k_{1,j}} + \dots + c_{n,j}x_n^{k_{n,j}} \equiv b_j \pmod{p}, \quad j = 1, \dots, s,$$

where  $a, b_j, c_{i,j}, k_{i,j} \in \mathbb{Z}$ , with  $\gcd(ac_{i,j}, p) = 1$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, s$ , and  $3 \leq k_{i,1} < \dots < k_{i,s}$ .

---

2010 *Mathematics Subject Classification.* 11D79, 11K38.

*Key words and phrases.* Multivariate congruences, distribution of points.

The interest to the systems of congruences (2) and (3) stems from the work of Fouvry and Katz [5], where a particular case of the congruence (2) and just one congruence of the type (3) (that is, for  $s = 1$ ) with the same odd exponents  $k_{1,1} = \dots = k_{n,1} = k$  and  $b_1 = 0$  is given as an example of a variety to which one of their main general results applies. In particular, in this case and for  $k \geq 3$ ,  $b_1 = 0$  (and fixed non-zero coefficients) we see that [5, Theorem 1.5] gives an asymptotic for the number of solutions with  $1 \leq x_i \leq h$ ,  $i = 1, \dots, n$ , starting from the values of  $h$  of size about  $\max\{p^{1/2+1/n}, p^{3/4}\} \log p$ . Here we show that a different and more specialised treatment allows to significantly lower this threshold, which now in some cases reaches  $p^{1/4+\kappa}$  for any  $\kappa > 0$ . Furthermore, this applies to the systems (2) and (3) in full generality and is uniform with respect to the coefficients.

More precisely, we use a combination of

- the bound of mixed character sums due to Chang [3];
- the result of Ayyad, Cochrane, and Zheng [1] on the fourth moment of short character sums;
- the bound of Wooley [14] on exponential sums with polynomials.

We note that the classical Pólya-Vinogradov and Burgess bounds of multiplicative character sums (see [6, Theorems 12.5 and 12.6]) in a combination with a result of Ayyad, Cochrane, and Zheng [1], has been used in [9, 10] to study the distribution of the single congruence (2) in very small boxes), and thus go below the  $p^{1/2}$ -threshold.

Here we show that the recent result of Chang [3] enables us now to study a much more general case of the simultaneous congruences (2) and (3).

Throughout the paper, the implied constants in the symbols “ $O$ ” and “ $\ll$ ” can depend on the degrees  $k_{i,j}$  in (2) and (3) as well as, occasionally, of some other polynomials involved. We recall that the expressions  $A \ll B$  and  $A = O(B)$  are each equivalent to the statement that  $|A| \leq cB$  for some constant  $c$ .

## 2. CHARACTER AND EXPONENTIAL SUMS

Let  $\mathcal{X}_p$  be the set of multiplicative characters modulo  $p$  and let  $\mathcal{X}_p^* = \mathcal{X}_p \setminus \{\chi_0\}$  be the set of non-principal characters. We also denote

$$\mathbf{e}_p(z) = \exp(2\pi iz/p).$$

We appeal to [6] for a background on the basic properties of multiplicative characters and exponential functions, such as orthogonality.

The following bounds of exponential sums twisted with a multiplicative character has been given by Chang [3] for sum in arbitrary finite fields but only for intervals starting at the origin. However, a simple examination of the argument of [3] reveals that this is not important for the proof:

**Lemma 1.** *For any character  $\chi \in \mathcal{X}_p^*$ , a polynomial  $F(X) \in \mathbb{Z}[X]$  of degree  $k$  and any integers  $u$  and  $h \geq p^{1/4+\kappa}$ , we have*

$$\sum_{x=u+1}^{u+h} \chi(x) \mathbf{e}_p(F(x)) \ll hp^{-\eta},$$

where

$$\eta = \frac{\kappa^2}{4(1+2\kappa)(k^2+2k+3)}.$$

We note that we do not impose any conditions on the polynomial  $F$  in Lemma 1.

On the other hand when  $\chi = \chi_0$ , we use the following a very special case of the much more general bound of Wooley [14] that applies to polynomials with arbitrary real coefficients.

**Lemma 2.** *For any polynomial  $F(X) \in \mathbb{Z}[X]$  of degree  $k > 2$  with the leading coefficient  $a_k \not\equiv 0 \pmod{p}$ , and any integers  $u$  and  $h$  with  $h < p$ , we have*

$$\sum_{x=u+1}^{u+h} \mathbf{e}_p(F(x)) \ll h^{1-1/2k(k-2)} + h^{1-1/2(k-2)} p^{1/2k(k-2)}.$$

Clearly Lemma 2 is nontrivial only for  $h \geq p^{1/k}$  which is actually the best possible range. Furthermore, in a slightly shorter range we have:

**Corollary 3.** *For any polynomial  $F(X) \in \mathbb{Z}[X]$  of degree  $k > 2$  with the leading coefficient  $a_k \not\equiv 0 \pmod{p}$ , and any integers  $u$  and  $h$  with  $p^{1/(k-1)} \leq h < p$ , we have*

$$\sum_{x=u+1}^{u+h} \mathbf{e}_p(F(x)) \ll h^{1-1/2k(k-2)}.$$

We make use of the following estimate of Ayyad, Cochrane and Zheng [1, Theorem 1].

**Lemma 4.** *Uniformly over integers  $u$  and  $h \leq p$ , the congruence*

$$x_1 x_2 \equiv x_3 x_4 \pmod{p}, \quad u+1 \leq x_1, x_2, x_3, x_4 \leq u+h,$$

*has  $h^4/p + O(h^2 p^{o(1)})$  solutions as  $h \rightarrow \infty$ .*

We note that Lemma 4 is essentially a statement about the fourth moment of short character sums, see [1, Equation (4)]. In fact, the next result makes it clearer:

**Corollary 5.** *Let  $\rho(x)$  be an arbitrary complex valued function with*

$$|\rho(x)| \leq 1, \quad x \in \mathbb{R}.$$

*Uniformly over integers  $u$  and  $h \leq p$ , we have*

$$\sum_{\chi \in \mathcal{X}_p} \left| \sum_{x=u+1}^{u+h} \rho(x) \chi(x) \right|^4 \leq h^4 + O(h^2 p^{1+o(1)}),$$

*as  $h \rightarrow \infty$ .*

*Proof.* Expanding the fourth power, and changing the order of summation, we obtain

$$\begin{aligned} & \sum_{\chi \in \mathcal{X}_p} \left| \sum_{x=u+1}^{u+h} \rho(x) \chi(x) \right|^4 \\ &= \sum_{\chi \in \mathcal{X}_p} \sum_{x_1, \dots, x_4=u+1}^{u+h} \rho(x_1) \rho(x_2) \rho(x_3) \rho(x_4) \chi(x_1 x_2 x_3^{-1} x_4^{-1}) \\ &= \sum_{x_1, \dots, x_4=u+1}^{u+h} \rho(x_1) \rho(x_2) \rho(x_3) \rho(x_4) \sum_{\chi \in \mathcal{X}_p} \chi(x_1 x_2 x_3^{-1} x_4^{-1}). \end{aligned}$$

Using the orthogonality of characters, we write

$$\begin{aligned} & \sum_{\chi \in \mathcal{X}_p} \left| \sum_{x=u+1}^{u+h} \rho(x) \chi(x) \right|^4 \\ &= \sum_{\chi \in \mathcal{X}_p} \sum_{x_1, \dots, x_4=u+1}^{u+h} \rho(x_1) \rho(x_2) \rho(x_3) \rho(x_4) \chi(x_1 x_2 x_3^{-1} x_4^{-1}) \\ &= (p-1) \sum_{\substack{x_1, \dots, x_4=u+1 \\ x_1 x_2 \equiv x_3 x_4 \pmod{p}}}^{u+h} \rho(x_1) \rho(x_2) \rho(x_3) \rho(x_4) \\ &\leq (p-1) \sum_{\substack{x_1, \dots, x_4=u+1 \\ x_1 x_2 \equiv x_3 x_4 \pmod{p}}}^{u+h} 1. \end{aligned}$$

Using Lemma 4 we derive the desired bound.  $\square$

## 3. MAIN RESULT

We are now able to present our main result. Let  $\mathfrak{B}$  be a cube of the form

$$\mathfrak{B} = [u_1 + 1, u_1 + h] \times \dots \times [u_n + 1, u_n + h]$$

with some integers  $h, u_i$  with  $1 \leq u_i + 1 < u_i + h < p$ ,  $i = 1, \dots, n$ . We denote by  $N(\mathfrak{B})$  the number of integer vectors

$$(x_1, \dots, x_n) \in \mathfrak{B}$$

satisfying (2) and (3) simultaneously.

As we have mentioned the case of just one congruence (2) has been considered in [9, 10], so we always assume that  $s \geq 1$  (and thus  $n \geq 3$ ).

Let

$$k = \min\{k_{i,j} : i = 1, \dots, n, j = 1, \dots, s\},$$

$$K = \max\{k_{i,j} : i = 1, \dots, n, j = 1, \dots, s\}.$$

**Theorem 6.** *For any fixed  $\kappa > 0$  and*

$$p > h \geq \min\{p^{1/4+\kappa}, p^{1/(k-1)}\}$$

*we have*

$$N_p(\mathfrak{B}) = \frac{h^n}{p^{s+1}} + O\left(h^n p^{-1-\eta(n-4)} + h^{n-2} p^{-\eta(n-4)}\right),$$

*where*

$$\eta = \frac{\kappa^2}{4(1+2\kappa)(K^2+2K+3)}.$$

*Proof.* Using the orthogonality of characters, we write

$$N_p(\mathfrak{B}) = \sum_{(x_1, \dots, x_n) \in \mathfrak{B}} \frac{1}{p^s} \sum_{\lambda_1, \dots, \lambda_s=0}^{p-1} \mathbf{e}_p \left( \sum_{j=1}^s \lambda_j \left( \sum_{i=1}^n c_{i,j} x_i^{k_{i,j}} - b_j \right) \right) \frac{1}{p-1} \sum_{\chi \in \mathcal{X}_p} \chi(x_1 \dots x_n a^{-1}).$$

Hence, changing the order of summation, we obtain

$$N_p(\mathfrak{B}) = \frac{1}{(p-1)p^s} \sum_{\lambda_1, \dots, \lambda_s=0}^{p-1} \mathbf{e}_p \left( - \sum_{j=1}^s \lambda_j b_j \right) \sum_{\chi \in \mathcal{X}_p} \chi(a^{-1}) \prod_{i=1}^n S_i(\chi; \lambda_1, \dots, \lambda_s),$$

where

$$S_i(\chi; \lambda_1, \dots, \lambda_s) = \sum_{x=u_i+1}^{u_i+h} \mathbf{e}_p \left( \sum_{j=1}^s \lambda_j c_{i,j} x^{k_{i,j}} \right), \quad i = 1, \dots, n.$$

Separating the term  $h^n/(p-1)p^s$ , corresponding to  $\chi = \chi_0$  and  $\lambda_1 = \dots = \lambda_s = 0$ , we derive

$$(4) \quad N_p(\mathfrak{B}) - \frac{h^n}{(p-1)p^s} \ll \frac{1}{p^{s+1}} (R_1 + R_2),$$

where

$$\begin{aligned} R_1 &= \sum_{\lambda_1, \dots, \lambda_s=0}^{p-1} \sum_{\chi \in \mathcal{X}_p^*} \prod_{i=1}^n |S_i(\chi; \lambda_1, \dots, \lambda_s)|, \\ R_2 &= \sum_{\substack{\lambda_1, \dots, \lambda_s=0 \\ (\lambda_1, \dots, \lambda_s) \neq (0, \dots, 0)}}^{p-1} \prod_{i=1}^n |S_i(\chi_0; \lambda_1, \dots, \lambda_s)|. \end{aligned}$$

To estimate  $R_1$  we use Lemma 1 and write

$$R_1 \leq h^{n-4} p^{-\eta(n-4)} \sum_{\lambda_1, \dots, \lambda_s=0}^{p-1} \sum_{\chi \in \mathcal{X}_p^*} \prod_{i=1}^4 |S_i(\chi; \lambda_1, \dots, \lambda_s)|.$$

Using the Hölder inequality and Corollary 5, we obtain

$$\begin{aligned} \sum_{\chi \in \mathcal{X}_p^*} \prod_{i=1}^4 |S_i(\chi; \lambda_1, \dots, \lambda_s)| &\leq \left( \prod_{i=1}^4 \sum_{\chi \in \mathcal{X}_p^*} |S_i(\chi; \lambda_1, \dots, \lambda_s)|^4 \right)^{1/4} \\ &\ll h^4 + h^2 p^{1+o(1)}. \end{aligned}$$

Therefore,

$$(5) \quad R_1 \ll h^n p^{s-\eta(n-4)} + h^{n-2} p^{s+1-\eta(n-4)}.$$

Furthermore, for  $R_2$  we use Corollary 3 to derive

$$R_2 \leq h^{(n-2)(1-1/2K(K-2))} \sum_{\substack{\lambda_1, \dots, \lambda_s=0 \\ (\lambda_1, \dots, \lambda_s) \neq (0, \dots, 0)}}^{p-1} \prod_{i=1}^2 |S_i(\chi; \lambda_1, \dots, \lambda_s)|.$$

Using the Hölder inequality and the orthogonality of exponential functions (similarly to the proof of Corollary 5), we obtain

$$\begin{aligned} \sum_{\substack{\lambda_1, \dots, \lambda_s=0 \\ (\lambda_1, \dots, \lambda_s)=(0, \dots, 0)}}^{p-1} \prod_{i=1}^2 |S_i(\chi; \lambda_1, \dots, \lambda_s)| \\ \leq \left( \prod_{i=1}^2 \sum_{\lambda_1, \dots, \lambda_s=0}^{p-1} |S_i(\chi; \lambda_1, \dots, \lambda_s)|^2 \right)^{1/2} \ll p^s h. \end{aligned}$$

Thus

$$(6) \quad R_2 \ll h^{n-1-(n-2)/2K(K-2)} p^s.$$

Substituting the bounds (5) and (6) in (4) we obtain

$$\begin{aligned} N_p(\mathfrak{B}) - \frac{h^n}{p^{s+1}} \\ \ll h^n p^{-1-\eta(n-4)} + h^{n-2} p^{-\eta(n-4)} + h^{n-1-(n-2)/2K(K-2)} p^{-1}. \end{aligned}$$

Clearly,

$$\eta < \frac{1}{2K(K-2)}.$$

Thus we see that the second term always dominates the third term and the result follows.  $\square$

#### 4. COMMENTS

Clearly, for any  $\kappa > 0$ ,  $k \geq 5$  and  $p > h \geq p^{1/4+\kappa}$ , Theorem 6 implies that

$$N_p(\mathfrak{B}) = (1 + o(1)) \frac{h^n}{p^{s+1}},$$

as  $p \rightarrow \infty$ , provided that

$$n \geq (s + 1/2)\eta^{-1} + 4.$$

For  $k = 3$  and  $4$  the range of Theorem 6 becomes  $h \geq p^{1/2}$  and  $h \geq p^{1/3}$ . However it is easy to see that using the full power of Lemma 2 instead of Corollary 3 one can derive nontrivial results in a wider range. Namely, for any  $\kappa > 0$  there exists some  $\gamma > 0$  (independent on  $n$  and other parameters in (2) and (3)) such that, for  $h \geq p^{1/3+\kappa}$  if  $k = 3$  and for  $h \geq p^{1/4+\kappa}$  if  $k = 4$ , we have

$$N_p(\mathfrak{B}) = \frac{h^n}{(p-1)p^s} + O(h^{(1-\gamma)n}).$$

We also recall that for polynomials of small degrees stronger values of Lemma 2 are available, see [2] and references therein.

Note that the same method can be applied (with essentially the same results) to the systems of congruences where instead of (2) we have a more general congruence

$$x_1^{m_1} \dots x_n^{m_n} \equiv a \pmod{p}$$

for some integers  $m_i$  with  $\gcd(m_i, p-1) = 1$ ,  $i = 1, \dots, n$ .

Moreover, we recall that the Weil bound [13, Appendix 5, Example 12] (see also [7, Chapter 6, Theorem 3]) and the standard reduction between complete and incomplete sums (see [6, Section 12.2]) implies that

$$\sum_{x=u+1}^{u+h} \chi(G(x)) \mathbf{e}_p(F(x)) \ll p^{1/2} \log p,$$

where  $G(x)$  is a polynomial that is not a perfect power of any other polynomial in the algebraic closure  $\overline{\mathbb{F}}_p$  of the finite field of  $p$  elements. Thus for  $h \geq p^{1/2+\kappa}$ , using this bound instead of Lemma 1 allows us to replace (2) with the congruence

$$G_1(x_1) \dots G_n(x_n) \equiv a \pmod{p}$$

for arbitrary polynomials  $G_1(X), \dots, G_n(X) \in \mathbb{Z}[X]$  such that their reductions modulo  $p$  are not perfect powers in  $\overline{\mathbb{F}}_p$ . In fact, even for  $G_1(X) = \dots = G_n(X) = X$  (that is, for the congruence (2)) this leads to a result, which is sometimes stronger than those of [5] and Theorem 6.

## 5. ACKNOWLEDGMENT

The author is very grateful to Mei-Chu Chang for the confirmation that the main result of [3] applies to intervals in an arbitrary position.

This work was supported in part by the ARC Grant DP1092835.

## REFERENCES

- [1] A. Ayyad, T. Cochrane, and Z. Zheng, ‘The congruence  $x_1x_2 \equiv x_3x_4 \pmod{p}$ , the equation  $x_1x_2 = x_3x_4$  and the mean value of character sums’, *J. Number Theory*, **59** (1996), 398–413.
- [2] K. D. Boklan, and T. D. Wooley, ‘On Weyl sums for smaller exponents’, *Funct. et Approx. Comment. Math.*, **46** (2012), 91–107.
- [3] M.-C. Chang, ‘An estimate of incomplete mixed character sums’, *An Irregular Mind*, Bolyai Society Math. Studies, vol. 21, Springer, Berlin, 2010, 243–250.
- [4] É. Fouvry, ‘Consequences of a result of N. Katz and G. Laumon concerning trigonometric sums’, *Israel J. Math.*, **120** (2000), 81–96.
- [5] É. Fouvry and N. Katz, ‘A general stratification theorem for exponential sums, and applications’, *J. Reine Angew. Math.*, **540** (2001), 115–166.
- [6] H. Iwaniec and E. Kowalski, *Analytic number theory*, Amer. Math. Soc., Providence, RI, 2004.



- [7] W.-C. W. Li, *Number theory with applications*, World Scientific, Singapore, 1996.
- [8] W. Luo, ‘Rational points on complete intersections over  $\mathbb{F}_p$ ’, *Internat. Math. Res. Notices*, **1999** (1999), 901–907.
- [9] I. E. Shparlinski, ‘On the distribution of points on multidimensional modular hyperbolas’, *Proc. Japan Acad. Sci., Ser.A*, **83** (2007), 5–9.
- [10] I. E. Shparlinski, ‘On a generalisation of a Lehmer problem’, *Math. Zeitschrift*, **263** (2009), 619–631.
- [11] I. E. Shparlinski and A. N. Skorobogatov, ‘Exponential sums and rational points on complete intersections’, *Mathematika*, **37** (1990), 201–208.
- [12] A. N. Skorobogatov, ‘Exponential sums, the geometry of hyperplane sections, and some Diophantine problems’, *Israel J. Math.*, **80** (1992), 359–379.
- [13] A. Weil, *Basic number theory*, Springer-Verlag, New York, 1974.
- [14] T. D. Wooley, ‘Vinogradov’s mean value theorem via efficient congruencing, II’, *Preprint* 2011, (available from <http://arxiv.org/abs/1112.0358>).

DEPARTMENT OF COMPUTING, MACQUARIE UNIVERSITY, SYDNEY, NSW 2109,  
AUSTRALIA

*E-mail address:* `igor.shparlinski@mq.edu.au`